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PHILIP HALL'S 'RATHER CURIOUS' FORMULA FOR ABELIAN p-GROUPS

BY

AVINOAM MANN*

Institute of Mathematics, The Hebrew University of Jerusalem
Givat Ram, Jerusalem 91904, Israel
e-mail: mann@math.huji.ac.il

To the memory of my great teacher and friend, Shimshon Amitsur

ABSTRACT

We provide a new proof for a formula of Philip Hall concerning the orders of finite abelian groups.

'Rather curious' is Hall's understated description of the following result [Ha38]:

THEOREM: *Let p be a prime. Then*

$$\sum \frac{1}{|G|} = \sum \frac{1}{|\mathrm{Aut}(G)|} \tag{1}$$

where the summation ranges over all finite abelian p-groups.

Hall proves this by first noting that the left hand side is equal to $\sum p(n)p^{-n}$, where $p(n)$ is the partition function (hence the sum converges), then enumerating partitions in a certain way, and applying an explicit expression for $|\mathrm{Aut}(G)|$. Another proof is given by Macdonald [Mc], which needs first the development of the formalism of 'Hall algebras'. Here we offer yet another proof. It uses a partition identity of Euler, and is based on ideas from the area of subgroup growth, i.e. counting finite index subgroups. See [Lu95] or [MS].

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Let F be a free abelian group of rank n. We consider subgroups H of F such that F/H is a finite p-group satisfying $d(F/H) = n$. Here we denote by $d(G)$ the minimal number of generators of the group G. If G is a finite abelian p-group, then $d(G)$ is given by $p^{d(G)} = |G/G^p|$. Since $|F\colon F^p| = p^n$, the assumptions on H are equivalent to $|F\colon H|$ being a power of p, and $H \leq F^p$. Therefore the number of those subgroups H in our family which have index p^{n+k} in F is equal to the number of subgroups of index p^k in F^p, and this is equal to the number of subgroups of the same index in F, because $F^p \cong F$. Let $a_k =: a_k(F)$ be this number. We write $f(s) = \sum a_k(p^{-k})^s$. Then it is known that

$$f(s) = (1 - 1/p^s)^{-1}(1 - 1/p^{s-1})^{-1} \cdots (1 - 1/p^{s-n+1})^{-1}. \tag{2}$$

See Theorem 2.1 of [Lu93] or [Lu95]. ([Lu93] contains four proofs of (2), including references to the original papers. Another proof is given in [Ma], eq. (Z) on p. 444.)

We now count the subgroups H according to their factor groups $G = F/H$. Given an abelian p-group G, the number of subgroups $H \leq F$ such that $F/H \cong G$ is equal to $|\operatorname{Epi}(F,G)|/|\operatorname{Aut}(G)|$ ([Ha36], 1.4). Here $\operatorname{Epi}(F,G)$ denotes the set of homomorphisms of F onto G, and $|\operatorname{Epi}(F,G)|$ is equal to the number of ordered n-tuples that generate G. Now an n-tuple generates G iff its images in G/G^p generate this factor group. Assuming $d(G) = n$, we have that G/G^p is elementary abelian of order p^n, and the number of n-tuples generating it is $(p^n - 1)(p^n - p) \cdots (p^n - p^{n-1})$. Since $|G| = p^{n+k}$, we have that $|G^p| = p^k$, so the number of n-tuples generating G is $(p^n - 1)(p^n - p) \cdots (p^n - p^{n-1})p^{nk}$. We thus have

$$a_k(p^{-k})^n = (p^n - 1)(p^n - p) \cdots (p^n - p^{n-1}) \sum \frac{1}{|\operatorname{Aut}(G)|} \tag{3}$$

the summation being over all groups G of order p^{n+k} such that $d(G) = n$. Summing over all k, the left hand side becomes $f(n)$, so, using (2), we obtain

$$\frac{p^{-n^2}}{(1 - 1/p^n)^2(1 - 1/p^{n-1})^2 \cdots (1 - 1/p)^2} = \sum \frac{1}{|\operatorname{Aut}(G)|} \tag{4}$$

where this time the summation is over all abelian p-groups satisfying $d(G) = n$. Finally, we sum this expression over all n, and substitute the value $x = 1/p$ in the following identity of Euler ([HW], Theorem 351)

$$\sum p(n)x^n = 1 + \sum \frac{x^{n^2}}{(1 - x)^2(1 - x^2)^2 \cdots (1 - x^n)^2}. \tag{5}$$

This yields $\sum p(n)p^{-n} = \sum 1/|\mathrm{Aut}(G)|$, proving Hall's formula.

Remark 1: It is instructive to compare our proof with Hall's. The proof of [HW], Theorem 351, enumerates partitions by the size of the largest square contained in their dot diagram. The enumeration of partitions that Hall employs in [Ha38], although more elaborate, also starts from this square. In that respect there is a similarity between Hall's proof and ours. More precisely, Hall associates with each partition another one, say $(n_1, \ldots, n_k)$, whose largest component n_1 is the abovementioned size of the largest square contained in the original partition. He then shows that $\sum p^*(n)p^{-n}$, where $p^*(n)$ is the number of partitions of n associated to $(n_1, \ldots, n_k)$, is equal to $1/|\mathrm{Aut}(G)|$, where G has invariants $(n_1, \ldots, n_k)$. Since $n_1 = d(G)$, this equality of Hall's is a refinement of (4).

Another point is that Hall's proof is mostly combinatorial, only at the end the values of $|G|$ and $|\mathrm{Aut}(G)|$ are inserted. Our proof is more group theoretical, and at least $\sum 1/|\mathrm{Aut}(G)|$ occurs in it naturally.

Remark 2: As noted in [Mc], a result similar to Theorem 1 holds, if we replace finite abelian groups by finite modules over a discrete valuation ring with a finite residue field (in particular, taking this ring to be the p-adic numbers, we have the case of abelian groups). To prove this version by our method, change F to a free module, and count the submodules of finite index. In the various formulas above we then have to replace p by q, the order of the residue field.

Remark 3: It is possible that, by starting from a different choice for the group F, or by considering a different family of quotients, similar identities may be obtained. See also [Lu93], 2.2, for a different application of subgroup growth to partition identities.

References

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